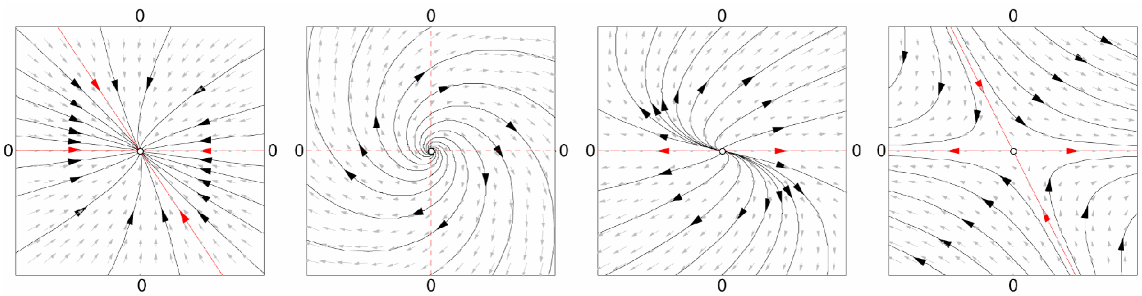
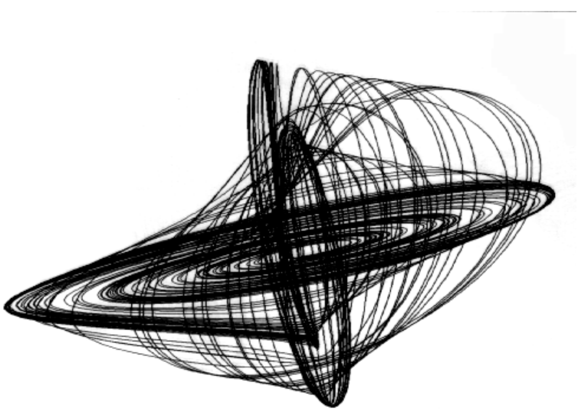
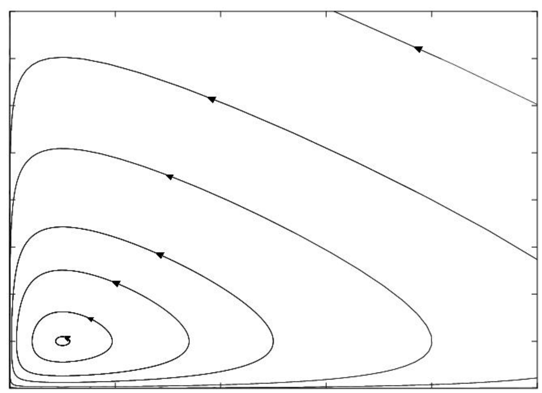
Dynamic Analysis of a System







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Part 1: Steady Points

1.1) Generalities

- What is a steady point?
- What is the property of the system at these particular points?
- What is the interest of finding these points?

1.2) Finding the Steady Points

- Write down the set of equations to solve to find them
- Resolution of the System
  - Turn equations into the simplest equivalent system possible
  PS: the new condition must be necessary and sufficient

  - If you can solve the new condition
    - write the expressions for the solutions
    - spell conditions of admissibility of these solutions if any

  - If you can't solve the new condition
    - determine how many potential solutions there are
    - give bounding of these solutions
    - spell conditions of admissibility of these solutions if any

- Physical Interpretation of your Results
  - Give qualitative interpretation of the influence of the model parameters
  - If you have condition of admissibility, give quantitative interpretation also

1.3) Example

Let's consider the following complex dynamic system (in its dimension-less form)

(Prey – Predator)

$$\begin{cases} \dfrac{dX}{dt} = \dfrac{aX}{1+X} - b\dfrac{XY}{b_0 + X} \\ \dfrac{dY}{dt} = c\dfrac{XY}{1+XY} - dY \end{cases}$$

To find the <u>positive</u> steady points of the system:
- Prove that (0,0) is a steady point
- Prove that the abcissae of the other steady points satisfy a quadratic equation
- Prove that all the positive roots of this equation yield an admissible steady point
- Solve the quadratic equation
  - How many positive roots are there?
  - Focus in particular on the various conditions on the model parameters.

Part 2: Dynamic Stability of the Steady Points

2.1) Generalities

- What is the Meaning of the Study of the Dynamic Stability?

- Why do we use the Jacobian and its Eigenvalues?

- What is the rule for stability?

- In 2 Dimensions
  - Describe Graphically the most important cases
  - Give the Simplified stability Criteria

2.2) Modus Operandi in 2D

Important: Each steady Point must be studied independently

Modus Operandi:

- Calculate the general expression Jacobian Matrix

- Calculate Jacobian at Steady Point
  - Use equations satisfied by steady Point to simplify the matrix
  - Insert coordinates of the Steady Point if you have them

- If the resulting expression of the Jacobian is simple (<u>Very Rare Case</u>)
  - Find the Eigenvalues
  - Analyze the sign of their real parts with regards to the parameters
  - Conclude on the stability of the point

- If the Jacobian is complex (Normal case)
  - Calculate the trace and determinant of the 2D matrix
  - Analyze their sign with regards to the parameters
  - Conclude on the stability of the point

2.3) Example

Let's consider the simple following Dynamic System

(Prey – Predator 2)

$$\begin{cases} \dfrac{dX}{dt} = \dfrac{aX}{a_0 + X} - bXY \\ \dfrac{dY}{dt} = cXY - dY \end{cases}$$

- Prove the system has two steady points

- Prove one is a stable point while the second is stable as long as the model parameters are strictly positive

Part 3: Studying the Vector Field (VF)

3.1) Generalities

- What is the Vector Field?

- What are tin important components of a VF?

- Make an analogy between sketching a VF and sketching a function

3.2) Sketching a Vector Field

The general allure of the Vector Field obviously depends on the behaviour of its Steady Points, on the possible existence of limit cycles, on potential behaviours at infinity… It is therefore liable to vary significantly with the model parameters.
A thorough study should have identified all the possible cases by the time you reach the sketching phase. Vector Fields must be sketched for all the corresponding possible cases

NB: The question of the existence of limit cycles and general behaviour at infinity is a complex question (see WIKI Document 2) that needs to be tackled prior to sketching the VF.

- Place the steady points with the behaviour of the flow at their vicinity
- Plot in the VF, dx/dt=0 and dy/dt=0.

Draw the general trend of the VF (use the sign the dx/dt and dy/dt)

- Sketch the behaviour at infinity of the trajectories
- If the system is simple this is enough
- If the system is more complex:
  - Computer simulations are needed to get a feel for the result
  - Simulate the VF for a few values of your parameters and initial values

3.3) Example

Sketch the VF for :

(Prey – Predator 2)

$$\begin{cases} \dfrac{dX}{dt} = \dfrac{aX}{a_0 + X} - bXY \\ \dfrac{dY}{dt} = cXY - dY \end{cases}$$

We will admit that the trajectories are bounded as the time goes to infinity